Green Function for a small step in Paramagnetic spin model

Yan Zhao

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Problem:
N spins forming a 1-D spin chain and they act independently.
σ_i is the i^{th} spin and σ_i can be $\begin{cases} 1 & up \\ 0 & down \end{cases}$
The configuration of the system is $\underset{\sim}{\sigma} = (\sigma_i, \sigma_2, \cdots, \sigma_N)$.
Master equation

$$\begin{aligned}
\frac{d}{dt} P_t(\underset{\sim}{\sigma}) &= -\sum_{\underset{\sim}{\sigma}''} W(\underset{\sim}{\sigma} \to \underset{\sim}{\sigma}'') P_t(\underset{\sim}{\sigma}) + \sum_{\underset{\sim}{\sigma}'} W(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}) P_t(\underset{\sim}{\sigma}') \\
&= -\sum_{\underset{\sim}{\sigma}'} \sum_{\underset{\sim}{\sigma}''} W(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}'') P_t(\underset{\sim}{\sigma}') \delta_{\underset{\sim}{\sigma}', \underset{\sim}{\sigma}} + \sum_{\underset{\sim}{\sigma}'} W(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}) P_t(\underset{\sim}{\sigma}') \\
&= \sum_{\underset{\sim}{\sigma}'} \left(W(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}) - \delta_{\underset{\sim}{\sigma}', \underset{\sim}{\sigma}} \sum_{\underset{\sim}{\sigma}''} W(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}'') \right) P_t(\underset{\sim}{\sigma}') \\
&= \sum_{\underset{\sim}{\sigma}'} L(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}) P_t(\underset{\sim}{\sigma}')
\end{aligned}$$

By definition,

$$W(\underset{\sim}{\sigma} \to \underset{\sim}{\sigma}) = 0 \tag{1}$$

and

$$W(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}) = \sum_{i=1}^{N} W_i(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}) \tag{2}$$

and

$$W_i(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}) = \left(c\alpha\delta_{\sigma_i 1}\delta_{\sigma_i' 0} + (1-c)\alpha\delta_{\sigma_i 0}\delta_{\sigma_i' 1}\right)\prod_{j\neq i}\delta_{\sigma_j\sigma_j'} \tag{3}$$

And we have

$$L(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}) = W(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}) - \delta_{\underset{\sim}{\sigma}',\underset{\sim}{\sigma}}\sum_{\underset{\sim}{\sigma}''} W(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}'') \tag{4}$$

Then we can substitute (2) and (3) into (4) , which will give us

$$L(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}) = \sum_{i=1}^{N}\left[c\alpha\delta_{\sigma_i' 0}(\delta_{\sigma_i 1} - \delta_{\sigma_i\sigma_i'}) + (1-c)\alpha\delta_{\sigma_i' 1}(\delta_{\sigma_i 0} - \delta_{\sigma_i\sigma_i'})\right]\prod_{j\neq i}\delta_{\sigma_i\sigma_i'} \tag{5}$$

Let's go back to the master equation, by setting $dt = \epsilon$

$$\frac{P_{t+\epsilon}(\underset{\sim}{\sigma}) - P_t(\underset{\sim}{\sigma})}{\epsilon} = \sum_{\underset{\sim}{\sigma}'} L(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma})P_t(\underset{\sim}{\sigma}')$$

Then

$$\begin{aligned} P_{t+\epsilon}(\underset{\sim}{\sigma}) &= P_t(\underset{\sim}{\sigma}) + \sum_{\underset{\sim}{\sigma}'} \epsilon L(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma})P_t(\underset{\sim}{\sigma}') \\ &= \sum_{\underset{\sim}{\sigma}'}\left[\delta_{\underset{\sim}{\sigma}\underset{\sim}{\sigma}'} + \epsilon L(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma})\right]P_t(\underset{\sim}{\sigma}') \end{aligned}$$

Compared with

$$P_{t+\epsilon}(\underset{\sim}{\sigma}) = \sum_{\underset{\sim}{\sigma}'} G(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma};\epsilon)P_t(\underset{\sim}{\sigma}')$$

So the green function for this small step $(t \to t+\epsilon)$ is

$$\begin{aligned} G(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma};\epsilon) &= \delta_{\underset{\sim}{\sigma}\underset{\sim}{\sigma}'} + \epsilon L(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}) \\ &= \sum_{i=1}^{N}\frac{1}{N}\delta_{\sigma_i\sigma_i'}\prod_{j\neq i}\delta_{\sigma_i\sigma_i'} + \epsilon L(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}) \\ &= \sum_{i=1}^{N}\left[\frac{1}{N}\delta_{\sigma_i\sigma_i'} + c\epsilon\alpha\delta_{\sigma_i' 0}(\delta_{\sigma_i 1} - \delta_{\sigma_i\sigma_i'}) + (1-c)\epsilon\alpha\delta_{\sigma_i' 1}(\delta_{\sigma_i 0} - \delta_{\sigma_i\sigma_i'})\right]\prod_{j\neq i}\delta_{\sigma_i\sigma_i'} \end{aligned}$$

If a and b are just 1 or 0, we can write down

$$\delta_{ab} = 1 + 2ab - a - b.$$

By using this trick, we can simplify the green function. Finally, we will have

$$\begin{aligned}
&G(\underset{\sim}{\sigma}' \to \underset{\sim}{\sigma}; \epsilon) \\
= &\sum_{i=1}^{N} \left[\left(\frac{1}{N} - c\epsilon\alpha \right) + \left(\frac{2}{N} - 2\epsilon\alpha \right) \sigma_i \sigma_i' - \left(\frac{1}{N} - 2c\epsilon\alpha \right) \sigma_i - \left(\frac{1}{N} - \epsilon\alpha \right) \sigma_i' \right] \\
&\prod_{j \neq i} (1 + 2\sigma_i \sigma_i' - \sigma_i - \sigma_i') \\
= &\sum_{i=1}^{N} (A + B\sigma_i \sigma_i' - C\sigma_i - D\sigma_i') \prod_{j \neq i} (1 + 2\sigma_i \sigma_i' - \sigma_i - \sigma_i')
\end{aligned}$$

where

$$\begin{aligned}
A &= \frac{1}{N} - c\epsilon\alpha \\
B &= \frac{2}{N} - 2\epsilon\alpha \\
C &= \frac{1}{N} - 2c\epsilon\alpha \\
D &= \frac{1}{N} - \epsilon\alpha
\end{aligned}$$

Now I am still trying to change the form of this green function so that I could easily calculate the green function for a given path. And probably I could use other tricks instead of $\delta_{ab} = 1 + 2ab - a - b$. Please tell me if you have any ideas or questions, we can discuss and I really appreciate it!

References

[1] R. van Zon and J. Schofield, J. Chem. Phys. 122, 194502 (2005)